General Relativity in a Fish Tank

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1 Introduction

Einstein’s theory of general relativity predicts that a gravitational field can warp space-time\cite{1}, creating a skewed path along which light will travel\cite{2}. Therefore, gravitational fields can bend light. Einstein's theory was confirmed in 1919, during a total solar eclipse, when a star whose light passed close by the darkened sun was observed at a slightly different location than usual. A similar effect takes place on a much smaller scale when light travels through an optical medium with a continuously changing index of refraction. Such a medium is called a GRIN (GRAdient INdex) medium. GRIN media exist in nature, for example in mirages, where heated air has a lower index than cooler more dense air. The French mathematician Fermat (1601 –1665) proved that the path light follows as it moves through different media is the one that requires the least possible overall time. Light behaves the same way in warped space-time, and thus a GRIN medium can be used to model such a curved path.

This project consisted of creating and analyzing such a GRIN medium made by adding a layer of water on top of a layer of corn syrup in a fish tank. The two liquids gradually diffuse into one another, creating a gradient mixture in which the refractive index varies smoothly in the vertical direction. A laser beam that is initially pointed into the tank at an angle above the horizontal will follow a smooth curve, reach a maximum height, and then turn downwards towards the direction of increasing refractive index.

Three methods were used to measure refractive indices of uniform and blended liquids, as described in Section 4. Two of these methods produced a profile of the index gradient as a function of height, and these independently-obtained results were in good agreement. In a separate phase of the research, a numerical simulation was created to predict the path of light in a GRIN medium with a known gradient profile.
2 General Relativity and Light

2.1 Warping Space-Time

In Newtonian physics gravity is explained as a force between two massive objects, according to the formula \( F_g = \frac{GMm}{d^2} \). However, Einstein said that small objects do not orbit due to some mystical “attraction” but instead travel through space that is warped by a larger object [1]. This does not suggest that Newton is wrong; it simply proposes a different way of visualizing the influence of gravity. The idea of a warp in space-time can be thought of as such: the stronger the gravitational influence of an object, the larger the warp and visa versa. It is useful to visualize gravity in this manner to be able to better understand the effects of gravity on light. Einstein’s theory of general relativity states that gravity creates warps in space time; these space time warps can bend the path of light[3].

2.2 Gravitational Lensing

A result of light being influenced by gravity is known as gravitational lensing. This effect has been and is being studied by many astronomers and physicists. The effects of large gravitational fields can lead to multiple imaging of a distant object, the creation of caustics of the image being lensed, or the bending of the path of an objects light creating the illusion that the light is propagating from a point that it is not. Thus, light can be bent in many different fashions by the effects of a gravitational field. In theory, when an object is in perfect alignment with its lens it creates what is called an Einstein ring[4]. This theoretical ring is a circular lensing effect that takes place when the object being lensed, the len’s center of gravity, and the observer all lie on the same axis. The ring is made of an infinite number if images of the object being lensed. In nature, strong lenses don’t create an infinite number
of images but they can create multiple images of the lensed object[5]. This effect is visually confusing, but does not have many attractive properties that can be used by astronomers and physicists. On the other hand, microlensing has a lot of astrophysical applications. Microlensing occurs when the lens is relatively small, has a large gravitational field, and does not emit a lot of light (i.e. brown dwarfs, black holes etc.))[6]. The reason these objects are so useful is because some of them have a dark matter components [7]. Understanding how those lenses work, and what they are made of, can lead to defining a dark matter component of that area of space.

2.3 Refraction

The speed of light in a vacuum is $3 \times 10^8 m/s$. However, the speed of light decreases with respect to the medium through which it is traveling. For example, the speed of light in water is $2.25 \times 10^8 m/s$. The ratio of the speed of light in a vacuum, and the speed of light in a material is the index of refraction of that material[8]. The index of refraction of water can be calculated to be 1.33 from equation 1 below.

$$n = \frac{c}{v}$$  \hspace{1cm} (1)

Where $n$ is the index of refraction of some material, $c$ is the speed of light in a vacuum, and $v$ is the speed of light in that material.

When light travels from one medium to another at an angle, it refracts at an angle that is related to the two media’s indices of refraction. When a pencil is stuck into a cup of water, it appears to bend at an angle into the water. However, the pencil does not bend, the light reflecting off the pencil refracts at an angle through the water which creates the illusion that the pencil is bending. Snell’s law of refraction is a formula used to calculate the angle at
which light refracts when passing from one medium to another; see figure 1 and equation 2 below:

![Snell's Law Diagram](image_url)

Figure 1: Snell’s Law diagram

\[ n_i \sin(\theta_i) = n_t \sin(\theta_t) \]  

(2)

Where \( n_i \) is the index of refraction of the first medium, \( \theta_i \) is the angle at which the incident beam is introduced to the second medium relative to the normal (or the plane perpendicular to the interface), \( n_t \) is the index of refraction of the second medium, and \( \theta_t \) is the angle at which the transmitted light follows relative to the normal.

### 2.3.1 Polarized Light

Polarized light is discussed in this section since its properties were involved in measuring the varying index of refraction in the GRIN tank. All light has an electric and magnetic field. In the case of unpolarized light, the electric field oscillates at any angle and in any direction, however, linearly polarized light is created only when the electric field oscillates in a unidirectional line. The direction of that line denotes the direction of polarization is (p-polarization is in the parallel direction, and s-polarization is in the perpendicular direction)[9].
Figure 2: Relative intensity versus incident angle

Whenever light from the sun is scattered off of a surface, polarized light is created. Anyone who has worn polarized sun glasses realizes, through the reduction of reflected light, how abundant polarized light is. Polarized sunglasses block out light reflected off surfaces that is polarized, thus reducing glare. Polarized light has many attractive properties that make it light source that is often used in optical experiments. For exam Its relative intensities can be related to the index of refraction off which it is being reflected. It is also possible to make the intensity of the reflection of polarized light go to zero.

When the reflection of suitably polarized light is reduced to zero at an interface, the angle between the beam and the normal to the interface is at what is known as Brewster’s Angle. In other words, all of the light is being transmitted into the second medium of the interface. Brewster’s angle is very useful for measuring the index of refraction of different materials, such as glass. Shining a laser beam, into an interface between air and glass, at an angle, and adjusting that angle until the reflection goes to zero yields the refractive index of glass if that angle can be measured. Figure 2 below relates the relative intensity of the different types of polarized light reflected off an interface of two media of indices $n = 1.00$ and $n = 2.00$ when introduced at various angles. Where the curve for $R_p$ goes to zero is Brewster’s angle for these two media.
The formula for Brewster’s angle is:

\[ \theta_b = \arctan \left( \frac{n_t}{n_i} \right) \]  

(3)

Where \( \theta_b \) is Brewster’s angle, \( n_t \) is the refractive index of the transient medium, and \( n_i \) is the refractive index of the incident medium.

### 2.3.2 Gradient Index Materials

When light enters an interface of two different media, it will either bend towards or away from the normal to that interface. If there are 3 different media of increasing refractive index, then light entering the first interface will bend toward the normal, enter the second interface and again bending toward the normal. Taking this a step further with 100 different media, or layers, means 99 interfaces over the same distance. The light bends towards the normal when passing through all of these layers, creating an optical light path that appears is a curve. Diffusing two media of different refractive indices into one another creates an infinite amount of layers over the same finite distance, or a GRIN medium. Since a this medium can manipulate light to form a curve, it is possible that the its properties can be applied to conceptually model how light behaves in a gravitational field.

### 3 Methods and Results

This experiment used three different methods for describing different properties of the GRIN tank. The first method measures the displacement of a laser beam as it passes at an angle through a tank containing uniform medium. It was used to measure the index of the water and syrup making up the GRIN medium. This method is not effective at measuring a GRIN medium.
The other two methods were used to measure the index of blended media as a function of vertical height, thus creating a index profile that could be used to predict the path of the laser beam. The first of these involves measuring the intensity of light reflected from the inside surface of the tank wall, at the particular angle (Brewster’s angle) where the reflection of suitably polarized light from the outside surface drops to zero. A complicated formula derived from Fresnel equations and Snell’s law was used to create a profile for the GRIN tank. The final method involved measuring the vertical drop of an initially horizontal beam as it crossed a much thinner, specially made tank.

3.1 Beam Displacement Method

The beam displacement method measured the refractive index of the two individual materials that made up the GRIN tank. A formula which relates Snell’s law and to amount a beam is displaced in the horizontal direction was used to evaluate the refractive index of these individual media. This method led to a better understanding of how the two materials effect the path of light.

3.1.1 Setup

This method involved setting up a fish tank (30cm x 15cm x 20.5cm) with parallel walls on a table that had set angles to which the tank could be rotated; see Figure 3 below:

The reason the tank walls need to be parallel is because of the necessity for the incident beam and the beam that leaves the tank need to be parallel. Applying Snell’s law it is evident that the incident angle creates a transient angle which becomes the incident angle for the interface where the beam exits the tank. If these walls are parallel, that angle produces a beam that is parallel to the original incident beam. However, if the walls are not parallel,
Figure 3: Diagram of beam displacement setup

the final transient beam would not be parallel to the incident beam. This means that the horizontal displacement \( d \) measurements would be extremely inaccurate.

Since it is not possible to measure the transient angle of the beam inside the tank, Snell’s law was used to evaluate it. Rearranging equation 2 to solve for \( \theta_t \) gives:

\[
\theta_t = \arcsin \left( \frac{n_i \sin(\theta_i)}{n_t} \right)
\]

To estimate \( \theta_t \) it was necessary to estimate the index of refraction of the medium of the tank. Using that estimated value it was possible to create a theoretical curve for the displacement of the beam \( d \). See figure 4, and equation 5 below:

The data collected from this method was compared to the curve created by formula 4 and fit to it using a least squares method. After matching the curve to the data, the index of refraction of the material inside the tank was known.

3.1.2 Beam Displacement Results

Originally this method produced results that were not satisfactory. The theoretical curve was best fit to an index of about \( n = 1.29 \); whereas the accepted value for the refractive index
Figure 4: Geometry of the tank

\[
\sin(\theta_i - \theta_t) = \frac{d}{z} \quad d = z \sin(\theta_i - \theta_t)
\]

and: \[\cos(\theta_t) = \frac{L}{z} \quad z = \frac{L}{\cos(\theta_t)}\]

therefore: \[d = \frac{L \sin(\theta_i - \theta_t)}{\cos(\theta_t)}\] (5)

of water is \( n = 1.33 \). Figure 5 is a graph of the first data set of horizontal displacement, \( d \), versus the angle the tank was rotated to \((90 - \theta_t)\).

Where the lines are the theoretical curves, and the marks are the recorded data. After some deliberation, the source of inaccuracy became clear; the instrument being used to record the data was too crude. Fixing the instrument, and taking another set of data led to a goodness of .009 to an index of \( n = 1.33 \). See figure 6.

To the naked eye, these two graphs do not look much different from one another. This is due to the fact that the graphs are fit to two different indices. Also, it is hard to pick up the minute differences that make each graph more or less accurate.
Figure 5: $n = 1.29$ (left)

Figure 6: $n = 1.33$ (right)

This method generated results for the index of refraction of water (1.33), and corn syrup (1.48) that were used in the following methods. This method also increased the understanding of how different indices effect the optical path of light from a laser beam, and laid down a firm foundation for a better understanding of the GRIN medium.

### 3.2 Reflective Coefficient Method

The reflective coefficient method involves measuring the profile of the GRIN medium with respect to vertical depth. The goal of this method was to relate the relative intensity of the reflected light to the index of refraction at that depth. Moving the tank in the vertical direction at 5mm increments created a profile of refractive index versus depth.

Originally this method was sought out to use the Brewster’s angle formula to figure out the index at varying depths. However, there are two interfaces that need to be taken into account; where the beam interacts with the outside of the tank wall from air, and where the beam interacts with the inside of the tank wall and the inner medium. Although the formula for Brewster’s angle was not directly used to evaluate the GRIN, it lead to the discovery that
the walls of the tank are made out of a glass with an index of refraction of 1.56 as opposed to a standard index of refraction of 1.50.

3.2.1 Fresnel equations

Fresnel equations were used to related the relative intensity of the reflected light from the inner interface to the index of refraction of the inner medium. The polarized laser used in this experiment was set up such that the light was polarized in the parallel direction. Figure 7 below shows the circumstances under which Fresnel equation could be used; equation 6 is Fresnel equation for reflected light polarized in the parallel direction (p-polarized light).

![Fresnel Diagram](image)

Figure 7: Fresnel Diagram

\[ r_{\parallel} = \left[ \frac{n_t \cos(\theta_i) - n_i \cos(\theta_t)}{n_t \cos(\theta_i) + n_i \cos(\theta_t)} \right]^2 \]  

(6)
Where \( r_\parallel \) is the fraction of the incident light that is reflected off the interface, \( n_t \) is the refractive index of the medium inside the tank, \( \theta_i \) is the angle to the normal at which the incident beam propagates, \( n_t \) is the refractive index of the tank wall, and \( \theta_t \) is the angle to the normal at which the beam is transmitted into the medium inside the tank.

The inability to measure \( \theta_t \) arises again. However, this time it is not possible to estimate \( n_t \) since it varies with depth. It is necessary to approach the problem directly. Applying formula 3 in replace of \( \theta_i \) in equation 6 yeilds:

\[
\tag{7}
\begin{align*}
r_\parallel &= \frac{\left[ n_t \cos(\theta_i) - n_t \cos \left( \arcsin \left( \frac{n_t \sin(\theta_i)}{n_t} \right) \right) \right]^2}{\left[ n_t \cos(\theta_i) + n_t \cos \left( \arcsin \left( \frac{n_t \sin(\theta_i)}{n_t} \right) \right) \right]^2} 
\end{align*}
\]

Now that \( \theta_i \) is taken care of, \( n_t \) needs to be isolated. Identifying the root of the problem as \( \cos \left( \arcsin \left( \frac{n_t \sin(\theta_i)}{n_t} \right) \right) \) it was necessary to figure out the relationship of \( \cos(\arcsin(u)) \). After applying calculus, it became evident that the \( \cos(\arcsin(u)) = \sqrt{1 - u^2} \). Applying this relationship to equation 7 (where \( u = \frac{n_t \sin(\theta_i)}{n_t} \)) yields:

\[
\tag{8}
\begin{align*}
r_\parallel &= \frac{\left[ n_t \cos(\theta_i) - n_t \sqrt{1 - \left( \frac{n_t \sin(\theta_i)}{n_t} \right)^2} \right]^2}{\left[ n_t \cos(\theta_i) + n_t \sqrt{1 - \left( \frac{n_t \sin(\theta_i)}{n_t} \right)^2} \right]^2} 
\end{align*}
\]

Equation 8 put the variables in a form that could be evaluated to figure out the index of refraction of the medium inside the tank at various depths.

### 3.2.2 Reflective Coefficient Setup

This method used the same tank as the beam displacement method, and a Melles Griot 12.5mW polarized laser. Setting up the tank such that the beam entered the first interface
at Brewster's angle for air and the tank wall assured that none of the light was being reflected off of the first interface, and that all of it was being transmitted into the tank wall to then reflect off of the second interface (inner tank wall and GRIN medium). See figure 8.

![Figure 8: First reflection goes to zero](image)

Gradually changing the height gave an even distribution of the refractive index versus depth in the tank.

### 3.2.3 Reflective Coefficient Results

Using equation 8 it became evident that the relationship between reflective coefficient for p-polarized light and index of refraction are inversely related. The data collected in this method led to a graph that described the refractive index profile of the GRIN tank. See figure 9
Figure 9: Reflective coefficient and refractive index vs. depth

This graph of refractive index versus depth, along with reflective coefficient versus depth shows the inverse relationship. As expected, the gradient increases with depth. However, any assumption that that the index would be linear was disproved by these results. The gradual increase in gradient in the top and bottom of the tank, surrounding a very steep gradient in the middle suggests a few things. The gradient mixture is not 100% diffused, and that there is a very steep gradient in the middle of the tank.

3.3 Continuous Refraction Method

The next phase of the project was to not just measure the refractive index at different levels in a GRIN tank, but to model the curvature of the beam as it travels through the GRIN medium. Mirages are nothing more than the light from the sky bending towards a higher index of refraction (cold air has a higher refractive index than relatively hotter air). As light from the sky gets close to the ground, it enters a medium that has a lower index of refraction than the medium it was just in. This causes some of the light rays from the sky to start to bend back upwards away from the ground and towards the medium of higher refractive
index. The effect of which is the apparent reflection of the sky off the ground. This method differs from the previous since its goal is to describe a higher order principle of the tank which is how light will act when introduced to a GRIN.

3.3.1 The Light Curve

The first part of this method involved figuring out what kind of shape the laser beam will create in a GRIN medium. The beam appeared to be parabolic, but had nothing but visual observation to back up the hypothesis. Setting up a completely theoretical tank with a linear GRIN medium that increases over 100 layers figured out that the light will travel a curved path. See figure 10.

![Light path diagram](image)

Figure 10: Light path in a linear gradient

The dashes represent the points where this theoretical beam interacts with each of the 99 interfaces, and the curved line is a parabola that lines up with the theoretical data. The fact that the parabola fits this data so well provides strong evidence that the optical path of the laser beam through a linearly distributed GRIN medium is that of a parabola.
3.3.2 Continuous Refraction Setup

This method used a different tank that was much thinner and longer than the tank the other methods used[10]. The tank needs to be thinner is because it measures a vertical displacement \( d_v \) that is created by how much the light curves at a certain depth in the tank; see figure 11 below.

![Continuous Refraction Setup Diagram](image)

**Figure 11: Continuous refraction setup**

The angle that the beam leaves the tank creates some vertical displacement \( d_v \) that is related to the gradient index at that depth. As the gradient increases, the displacement also increases, denoting that they are directly related.
3.3.3 Continuous Refraction Results

Analyzing the vertical displacement versus depth in the tank gave a great understanding of the gradient versus depth. See figure 12

\[
\begin{array}{c}
\text{Vertical Displacement vs. Depth} \\
\text{Integral of Gradient Values (a.u.)}
\end{array}
\]

Figure 12: Vertical displacement vs. depth (left)

Figure 13: Numerical integration of figure 12 (right)

Since a stronger gradient leads to a greater vertical displacement, this data reinforces the ideas that the reflection coefficient method discovered. Again, a steep gradient in the middle of the tank is very prevalent. However, this data represents the respective gradient indices versus depth. In theory, it is one degree of order lower than the data collected in the reflective coefficient method. However, figure 13 provides a profile of index versus depth. This curve is in good agreement with the index versus depth graph generated from the data collected in the reflective coefficient method. The graph generated by this method is relative, the initial index of refraction at the top of this medium is unknown. Therefore, the results of this method can be given some starting value refractive index from which the rest of the gradient can be extrapolated. This a much easier way to measure the profile of the GRIN tank.
4 Future Study

With the knowledge of how different indices of different media influence light, these experiments could lead to the creation of a conceptual model of the behavior of light around a gravitational field. For example, a spherical index gradient should bend light around its center, simulating how light would behave in the gravitational field of a black hole. Ultimately these properties can be related in mathematical simulations to gravitational lensing effects[2].

Qualitatively, experiments are being done to derive the curvature of light in a radially varying GRIN medium. Some of the results show that the beam will bend in towards the center of the circular medium, reach a point where the path of light is perpendicular to the radius at that point, then turn out and leave the medium. Snell’s law can be used to see that the light will bend towards the normal until the path is perpendicular to the normal (in a circular case the normal is a radius drawn the the center of the circle at that specific point) Figure 14 shows a sketch of what the path of light in a radial GRIN medium could look like.
Figure 14: Radial GRIN medium (index increases as radius decreases)

This sketch shows that the path of light can loop around dependent on the strength of the gradient. Creating a planar radial GRIN medium can lead to the creation of a spherical GRIN structure which can conceptually model how light curves in space.

5 Conclusion

The vast information derived from all of these methods has created a very firm understanding on how different materials can effect the path of a laser beam. The methods used were very effective measuring the different indices of the different materials; and creating profiles for the two GRIN tanks. The experiments being done now should create better relationships that lead to ultimately conceptually model the curvature of space-time.
References


[8] “*Refraction*”, http://hyperphysics.phy-astr.gsu.edu/hbase/geoopt/refr.htm


[10] Prof. Erland Graf, for lending us his thin tank. Stony Brook University.