Abstract

Frequently, when two objects like screen doors or chain-link fences overlap, wavy, intricate patterns appear to form on the surface. These patterns are an example of “moiré” effects. The word moiré derives from the French “moirer,” meaning “to produce a watered textile by weaving or pressing.” The connection to fabric is not a coincidence: moiré patterns have long been revered for the rippled effects they produce in cloths such as watered silk. Moiré patterns, aside from being beautiful to look at, also have great applicability in a wide range of fields: uses can be found in navigation, steganography (image hiding, visual cryptography), and counterfeit prevention, to name only a few. Moiré effects have even been employed to study the structure of graphene, one of frontier topics in science today. In the field of optics, the moiré effect can be combined with the Talbot self-imaging effect in diffraction to determine the focal length of a lens and to test if a beam is collimated (if the rays of light are parallel to each other, rather than divergent or convergent). Moiré-Talbot wavefront sensors have recently been applied to testing the refractive error of the eye during cataract surgery and offer several advantages over other techniques.

Our research had two distinct phases, one entirely mathematical and computational and the other largely experimental. In both phases we modeled moiré patterns and effects with the powerful software tool MATHEMATICA. In phase (I) we created interactive models by simulating different periodic gratings (lines, concentric ellipses, concentric circles, and arrays of dots). The user-controlled simulations serve as effective pedagogical tools for understanding the theory behind moiré patterns. We also modeled a quasiperiodic grating, based on the Fibonacci number sequence. In phase (II) we constructed a simplified moiré-Talbot collimation-testing instrument and recorded moiré fringe patterns at 40 closely-spaced positions of the collimating lens around the nominal focal position. These images were analyzed with ImageJ software to extract accurate information on the fringe tilt angle as a function of lens position (defocus). Finally a MATHEMATICA program was used to predict the expected tilt angle as a function of defocus. The simulation is in quite good agreement with the experimental results.
1 Introduction

When two tuning forks of slightly different pitch are struck at the same time, a distinct, undulating sound – a “beat note” – is created by the varying temporal overlap of the two separate notes. Outside of the musical realm beat “notes” are often created when a sound or electromagnetic wave is slightly shifted in frequency by the Doppler effect and then re-combined with the original wave. One familiar example is the use of reflected microwaves to measure speeding cars. Beat notes are just one example of “beat effects” in which a new, magnified pattern emerges from the varying overlap of two original patterns. Beat effects can also manifest themselves as striking visual patterns called “moiré patterns.” These patterns are a mathematically intriguing phenomenon with a wide range of applications, and the subject of this research.

Moiré patterns are frequently observed when two objects with repetitive designs, such as window screens or chain-link fences, overlap. When one pattern is tilted or separated with respect to the other, or has a different period, intricate light and dark patterns appear. Moiré patterns have been recognized and revered for millenia, and have many important applications. The word “moiré” is speculated to have originated from the Latin word “marmoreus,” “marbled,” before morphing into the English “mohair,” and finally the French verb “moirer,” meaning “to produce a watered textile by weaving or pressing” [1]. The resulting English noun “moiré” refers to the fabric produced by such a process. Moiré fabrics, such as watered silk, have been revered since the Middle Ages for the rippled “watermarks” created by their unique weave [2]. Many famous paintings, such as “Peter I of Russia,” showcase examples of moiré-like effects in cloth [3].

The many applications of moiré effects include interferometry [4], strain measurement and topography [5], collimation testing [6], and focal length measurement [7]. Even more surprising uses can be found in navigation [8], steganography [9], and counterfeit prevention [10]. Quite recently, moiré patterns have even aided researchers studying the structure of graphene, a material with rather exceptional properties, that was the subject of the 2010 Nobel Prize in Physics [11]. Moiré patterns seen in graphene magnify the structure of the carbon lattice and can be used to detect minute amounts of strain in graphene and other materials [12].

1Moiré-like effects can also have undesirable consequences, such as the creation of artifacts (“aliasing”) in digital images or sound recordings. Aliasing occurs so frequently in digital photographs that manufacturers of digital cameras have introduced “anti-aliasing filters” into their cameras to attenuate the effect of aliasing [13].
In this work, the moiré effect has been studied both theoretically, through mathematical simulations, and experimentally, through measurements and analysis in an important application, determining the curvature of an optical wavefront. Simulations and analysis were both carried out with MATHEMATICA, a powerful mathematical computing software [14]. Pursuant to the two different aspects of this work, this paper is divided into two major sections: the first describes the moiré simulations that were created (§2), and the second discusses the collimation test instrument that was set up, the experimental results obtained, and the theoretical model that explains them (§3).

It should be noted that the collimation test instrument relies on another fascinating optical phenomenon: the Talbot effect. The Talbot effect is a near-field diffraction effect: light shone through a grating forms a self-image of the grating at fixed distances from the grating. The size of these images relative to the grating (the image magnification) depends on the curvature of the wavefront incident on the grating, and this magnification can be determined by a moiré method. As discussed further in §3.2, the Talbot effect is just as interesting mathematically as the moiré effect, and also has numerous separate applications. Wavefront sensors based on the combined Talbot and moiré effects have many uses beyond collimation testing. In particular, Talbot-moiré instruments have recently been developed for testing the refractive error of the eye during cataract surgery [15]. This type of aberrometer has the advantage of a much greater range than the more conventional Shack-Hartmann wavefront sensors [16].

2 Simulating the Moiré Effect

We first explored a variety of moiré effects and ways to create them. In particular, we have experimented with gratings that are not linear or that are not strictly periodic (quasiperiodic). The simulations consisted of creating virtual gratings of different pitch and overlapping them using MATHEMATICA to study the resulting patterns. We were inspired by the existing simulations on the Wolfram Mathematica website [14]. Four different periodic gratings were employed: lines, concentric ellipses, concentric circles, and arrays of dots. A quasiperiodic grating was also studied. It consisted of an array of dots arranged like florets on a sunflower and was generated using Vogel’s formula (2004) [17]. All models were animated using the manipulate tool in MATHEMATICA to allow for user interaction.
2.1 Linear Gratings

Linear moiré is moiré created with linear gratings. This form of moiré was modeled first, as it is the most studied of all forms of moiré patterns. Linear moiré is an excellent visual example of the beat effect, as it can be created by overlapping two linear gratings of different periods. It can be shown [18] that if the first grating has a period $\lambda_1$ and the second, a period of $\lambda_2$, then the period of the resulting moiré fringes is

$$\lambda_{\text{beat}} = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}$$  \[1\]

This equation can be derived from the formula used to determine the beat frequency of two interfering sound waves, $f_{\text{beat}} = |f_1 - f_2|$. Indeed, if one imagines that the linear gratings are instead sine or cosine functions, with the dark lines as maxima and the light spaces as minima, it is simple to see how fundamentally linear moiré and beat frequencies created by two sound waves are related.

It can be shown [18] that the shape of the moiré fringes created by two gratings with period $\lambda_1$ and $\lambda_2$ follows the form

$$\cos(\phi_1(x, y) - \phi_2(x, y))$$  \[2\]

where $\phi_1(x, y)$ and $\phi_2(x, y)$ represent the grating functions of the two linear gratings. Then, if the two gratings are overlaid with an angle of $2\alpha$ between them, the grating functions $\phi_1(x, y)$ and $\phi_2(x, y)$ can be written as

$$\phi_1(x, y) = \frac{2\pi}{\lambda_1}(x \cos \alpha + y \sin \alpha)$$  \[3\]

$$\phi_2(x, y) = \frac{2\pi}{\lambda_1}(x \cos \alpha - y \sin \alpha)$$  \[4\]

From here, one can simplify Eqn. [2] by rewriting the argument in the cosine function as:

$$\phi_1(x, y) - \phi_2(x, y) = \frac{2\pi}{\lambda_{\text{beat}}}x \cos \alpha + \frac{4\pi}{\lambda}y \sin \alpha$$  \[5\]

where $\lambda$ is the average line spacing [18]. Now, the locations of the centers of the moiré fringes are given by

$$\phi_1(x, y) - \phi_2(x, y) = M2\pi$$  \[6\]

where $M$ is the fringe order (the integer corresponding to a fringe; ie, for the first fringe, $M = 1$). Creath
and Wyant [18] described two special cases for Eqn. [5]: when $\lambda_1 = \lambda_2 = \lambda$, and when $\alpha = 0$. The former case was studied first.

When the periods of the two gratings are the same, $\lambda_{\text{beat}}$ in Eqn. [1] is seen to become infinite, and thus the first term of Eqn. [5] will go to zero. Therefore, the $x$-component of Eqn. [5] is eliminated, and the shape of the resulting moiré fringes is dependent only on the $y$-component, namely:

$$\phi_1(x, y) - \phi_2(x, y) = \frac{4\pi}{\lambda} y \sin \alpha$$

One can also solve for this mathematically: when the periods of the gratings are the same, it follows that $\lambda$ is equal to $\lambda$. By setting Eqn. [7] equal to Eqn. [6] (solving for the locations of the fringe centers), one obtains the following equation:

$$M\lambda = 2y \sin \alpha$$

The case of $\lambda_1 = \lambda_2 = \lambda$ was modeled using MATHEMATICA. Two sets of parallel lines were plotted to represent the gratings. These sets of lines had different slopes to mimic the rotation of one grating with respect to the other. $\alpha$ can be found in terms of the slopes of the two lines

$$\tan(2\alpha) = \frac{m_1 - m_2}{1 + m_1m_2}$$

where $m_1$ and $m_2$ are the respective slopes of the two sets of parallel lines.

If the sets of parallel lines that comprise the two gratings are taken to follow the equations $y_1 = m_1x - ct$ and $y_2 = m_2x - ct$, respectively, where $t$ is the iterating value, then one can solve for the periods of the gratings. This is done by setting $y$ equal to zero and solving for $x$. It becomes apparent that, left as is, the two gratings will have different periods, as $\lambda_1 = \frac{ct}{m_1}$ and $\lambda_2 = \frac{ct}{m_2}$. This can be corrected by multiplying the $c$ value of $y_1$ by $\frac{m_1}{m_2}$. The model was animated with the manipulate function to allow the user to control the slopes of the two gratings, the scale of the plot, and the horizontal displacement of one grating with respect to the other. Figure 1 (a) shows a screenshot of the output of the MATHEMATICA code. The model displayed the expected horizontal fringes when one grating was tilted with respect to the other. In addition, there appeared to be horizontal fringes when $\lambda_1$ was an integral multiple of $\lambda_2$ (i.e. $\lambda_1 = a\lambda_2$). This was surprising, as Eqn. [1] would not go to infinity in this case and cause the first term of Eqn. [5] to vanish. A screenshot of the case of $\lambda_1 = 2\lambda_2$ is shown in Fig. 1 (b).
The second special case, when $\alpha = 0$, was also looked at briefly. Two gratings with a very slight difference in period were plotted by vertical lines on MATHEMATICA. Users move one grating horizontally in the negative or positive direction to see the resulting fringe shift. In the case where the gratings are not tilted with respect to each other, the second term of Eqn. [2] and the cosine of the first term vanish. The grating equation becomes:

$$\phi_1(x, y) - \phi_2(x, y) = \frac{2\pi}{\lambda_{\text{beat}}} x$$  \[10\]

By setting Eqn. [10] equal to Eqn. [6], one obtains the equation:

$$M \lambda_{\text{beat}} = x$$  \[11\]

Because there is no $y$-component to this equation, the resulting moiré fringes should not change as $y$ changes. This implies that the fringes observed are vertical, as seen in Fig. 1 (c).

### 2.2 Nonlinear Gratings

While linear moiré is fascinating, one must look at nonlinear gratings to see truly surreal moiré patterns. Two forms of nonlinear gratings were studied: gratings comprised of lines (circular, elliptical) and gratings

Figure 1: (a) Left: MATHEMATICA output showing a simulated moiré pattern for $\lambda_1 = \lambda_2$. (b) Center: The same, but for $\lambda_1 = 2\lambda_2$. The top panels in each figure allow the user to control, in order, the relative horizontal displacements of the two gratings ($c$), line slopes ($m_1, m_2$), the scales of the plots, and the integer ratio $a$ of $\lambda_1$ to $\lambda_2$. The case of $a = 2$ is shown in (b). Notice the horizontal fringes in each image. The axes scales are in arbitrary units. (c) Right: MATHEMATICA output showing a simulated moiré pattern for the case of $\alpha = 0$. The axes units are arbitrary. The period of the second grating (G2) is changed by the user. Note the vertical fringes, as expected from Eqn. [11]
comprised of arrays of dots.

Among the first to study moiré patterns created with nonlinear gratings was Oster et al. (1963) [19], in a paper that described the mathematics behind moiré patterns created with combinations of various linear and nonlinear gratings.

One of the cases that was studied by Oster et al. was that of two gratings of equispaced concentric circles. It was shown that when two gratings comprised of concentric circles are superimposed and shifted with respect to each other, the resulting fringes take the shape of hyperbolas with a focus at the center of each circle. When the gratings are further distanced, an ellipse forms between the hyperbolas. Motivated by these results, gratings comprising concentric circles were created using MATHEMATICA. As with the linear models, this model was animated, allowing users to change the number of concentric circles in each grating (thus changing the period of the grating) and the horizontal and vertical displacement of the gratings with respect to each other, as illustrated in the screenshot shown in Fig. 2 (a). It was seen that, indeed, the moiré patterns formed were in the shape of hyperbolas, and an ellipse was formed in the center when the gratings were moved sufficiently far apart.

From circular gratings, it was logical to create elliptical gratings and similarly animate the models. Two types of concentric elliptical gratings were created. In the first, the semi-major and semi-minor axes were

![Figure 2: (a) Left: MATHEMATICA output showing a simulated moiré pattern for gratings comprising concentric circles. (b) Center: The same, but for concentric ellipses. (c) Right: Moiré pattern for elliptical gratings where the semi-major axes are kept constant. Notice the markedly different patterns seen in (b) and (c). In each figure the panels at the top allow the user to control the scale factors of the ellipses and the displacements of the gratings.](image-url)
varied in a ratio that could be changed by the user, so that the resulting ellipses were truly concentric. In the second, the semi-major axis was kept the same, but the semi-minor axis was varied, creating a grating that looked like a cat’s eye. Figures 2 (b) and 2 (c) show the screenshots of the MATHEMATICA output for these cases. Although the two types of gratings themselves appeared to be rather similar, the moiré patterns created had a very discernible dissimilarity between them.

Another type of grating simulated was the dot array. This study was motivated by the discovery in the lab of several thin squares of metal with holes punched out in a regular rectangular pattern. When one such square was placed on top of another square and tilted, moiré patterns that looked like a magnified form of the holes punched out in the squares appeared as if from nowhere.

A similar dot array was modeled using MATHEMATICA. Two such gratings were created and overlapped using the Prolog function [14], and animated so that the user determined the number of dots in each grating. Horizontal and vertical shifts were added to center one grating with respect to the other.

These gratings were modified so that the user could change the horizontal or vertical scale of one grating with respect to the other. The resulting fringes were similarly elongated or compressed. Figure 3 (a) shows a screenshot of the MATHEMATICA output for gratings comprising dot arrays with a width to length ratio of 1. In Fig. 3 (b), the user has the option to vary the width to length ratio of the second grating. Moiré patterns created when one dot array is tilted with respect to the other are shown in Fig. 3 (c). The pattern observed is similar to that seen for the mesh gratings found in the lab, as shown in Fig. 3 (d).

The dots in Fig. 3 have rectangular packing, which affects the resulting moiré patterns. There are, however, other methods of packing dots in an array. One particularly space-efficient form is hexagonal packing, which is frequently seen in nature, such as in crystalline structures. For example, iron in FeO and Pt (111) are both packed hexagonally, which is more space-efficient than rectangular packing. When very thin layers of FeO and Pt(111) are overlapped, triangular-shapes moiré patterns can be observed [20].

A similar scenario was modeled by generating two dot arrays packed in a hexagonal manner. This is illustrated in Fig. 4 (a). Each of the dot arrays were formed by adding together two other identical arrays to form the final hexagonal grid, which was created as follows: First, a grid of $1 \times 2$ rectangles ($y$ is twice $x$ at any given dot) was modeled. Then, a replica of this grid was superimposed on the first and shifted along the $x$-axis and the $y$-axis by half the length of the respective sides. The moiré patterns were generated by compressing one grid equally in both axes. As expected, triangular moiré was observed. This can be compared to the triangular moiré created by the overlap of FeO and Pt(111) (Fig. 4(b)). The dot arrays were
similar to linear gratings in that a greater difference in period (a greater compression) created more moiré patterns in a given area.

Figure 3: MATHEMATICA output showing a simulated moiré pattern for several different dot arrays. (a) Top left: gratings with width to length ratio of 1. (b) Top right: width to length ratio of the second grating variable. The figure shows the particular case of 9:8. (c) Bottom left: Tilted gratings. (d) Bottom right: Grating found in the lab. The grating is shown to scale. Note the similarity between the moiré patterns observed here and those observed in the bottom left image (c).
2.3 Quasiperiodic Gratings

All the gratings described so far are periodic in some shape or form, but moiré patterns do not necessarily have to be formed by periodic gratings. Aperiodic (random) gratings can also create moiré patterns, called “Glass patterns” [21]. One can achieve aperiodicity by creating an array of randomly placed dots; Glass and Pérez [21] created aperiodic gratings by splattering paint on a transparency. If the random dot pattern is placed on top of its copy and tilted, moiré patterns can be observed.

It was more recently observed that Glass patterns can be made to have any shape [22]. If the dots in one random array are replaced with any shape, for example, the number “2;” and then placed under an identical, but slightly tilted array that has holes punched out instead of printed dots, the moiré patterns take on the shape of a single, enlarged “2”. This phenomenon extends to periodic arrays as well; however, in the case of periodically arranged dots, many smaller forms of “2” that repeat at regular intervals are seen, rather than a single large shape.

This fascinating effect prompted a question: if aperiodically arranged dots generate a single large form of the shape, and periodically arranged dots generate many smaller forms of the shape, what effect do quasiperiodically arranged dots have on Glass patterns?

The class of quasiperiodic patterns falls directly in between the class of random patterns and the class of periodic patterns. A quasiperiodic pattern is one that retains some structure, but lacks periodicity. This
means that it does not have “translational symmetry”– if one were to put one quasiperiodic structure on top of another identical to it and move the top structure from left to right, there would be no other place at which the structures would be in perfect alignment. In contrast, consider a perfectly periodic structure, such as a picket fence or a sine wave. The structures would be in alignment after every period.

Quasiperiodic structures were very recently brought to widespread notice when the 2011 Nobel Prize in Chemistry was awarded for the discovery of “quasicrystals” [23]. Quasicrystals, crystals that have quasiperiodic structures, were first observed in Aluminium-Manganese alloys in 1982, when the anomalous electron diffraction patterns produced by the alloys were noticed [24]. Many other compounds that have quasicrystalline structure have since been found. Indeed, the discovery of quasicrystals was so fundamental that the formal definition of a “crystal” had to be re-worded to include the concept of periodicity that had previously always been taken for granted.

Aside from quasicrystals, there is another famous example of a quasiperiodic structure– the Fibonacci sequence. This is generated by the rule: $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, $F_1 = 1$, $F_2 = 1$ The Fibonacci sequence appears remarkably often in nature. The fruitlets of a pineapple, the scales of a pine cone, and the florets of a sunflower are all examples of natural objects that follow the Fibonacci sequence [25].

We decided that before looking at quasiperiodically generated Glass patterns, a more general quasiperiodic grating would be modeled. This quasiperiodic grating would be used to create moiré patterns in the same way that periodic gratings were previously used. After deliberation on how to create a quasiperiodic

![Figure 5: (a) Left: MATHEMATICA output showing a simulated moire pattern for overlapping sunflower floret patterns which is an example of a quasiperiodic grating. The ratio of the two gratings in this example is 8:7. (b) Right: A photograph of a sunflower taken by the author (July 2012).]
grating, it was decided that the grating should in some way be connected to the Fibonacci sequence, as the Fibonacci sequence itself is quasiperiodic. This connection was found by simulating the arrangement of florets on a sunflower. The formula for the layout of florets on a sunflower has been found to rely on the golden angle, which in turn relies on the Fibonacci sequence, and was worked out by Vogel (1979) [17]. This makes the arrangement of florets on a sunflower quasiperiodic.

Vogel’s formula, in polar coordinates, can be written as \( \theta = \frac{2\pi}{\phi^2} n \), where \( r = c\sqrt{n} \), \( c \) is the scale factor, \( \phi \) is the golden ratio, and \( n \) is the counter. This equation was used to create a quasiperiodic grating to mimic a sunflower floret pattern. Figure 5 (a) shows the moiré pattern generated by overlapping two modeled sunflower florets. The index \( n \) takes the range \([1, 500]\) and the ratio of the two gratings is 8:7.

3 Collimation Test Instrument

The second major component of this research was assembling and studying a simplified instrument for testing the collimation (wavefront curvature) of a laser beam. This instrument provides an excellent example of the wide range of optical applications made possible by two fascinating optical phenomena: the moiré effect and the Talbot effect (described in §3.2 below). The overall goal was to model the setup mathematically in order to predict its sensitivity and accuracy.

3.1 Collimation

Collimation in optics refers to obtaining parallel rays from a divergent beam of light emitted by a source. In principle this can be achieved by placing a converging lens at its focal distance from the source; however, a beam will only be perfectly parallel forever if the source of light is a point source. In real life, it is not possible to have a point source, and thus the beam will eventually diverge. In this case, the definition of “collimation” becomes more nuanced. For the purposes of this discussion, a beam is considered collimated when the waves leaving the collimating lens are flat (planar), i.e., they have infinite radius of curvature. A high quality of laser beam collimation is often a necessity in optics research setups where great precision is required. These include the Fourier transform generator, the optical coherent processor, and the image transformer [26].

A simple method of determining whether a beam is collimated involves viewing the beam on a screen (a piece of paper) at various relatively large distances from the lens and adjusting the setup so that the spot size
does not change as the beam travels. This technique is both tedious and highly subjective, and an easier and more exact method is preferable. This challenge has been addressed by research over the past few decades on methods, instruments and devices for testing the collimation of laser beams with high precision.

3.2 Talbot Effect

The Talbot effect was first observed in 1836 by Henry Fox Talbot, one of the pioneers of photography. Although Talbot himself did not pursue his observations very far, Lord Rayleigh continued the study of the Talbot effect in 1881. The Talbot effect occurs when a light beam is shone through a grating; a self image of the grating is formed at fixed distances (multiples of the Talbot distance) from the original grating, so long as the grating is still in the near field. In the far field, the self image is replaced with the more familiar far-field (Fraunhofer) diffraction pattern.

Lord Rayleigh (1881) showed that the Talbot distance could be written as

$$Z_T = \frac{2d^2}{\lambda}$$

where $d$ is the period of the grating and $\lambda$ is the wavelength of the incident light [27]. It was later found by Cowley and Moodie (1957) [28] and Winthrop and Worthington (1965) [29] that there exist fractional Talbot images as well. For example, at half the Talbot distance, an image of the grating forms shifted by half the grating’s period. At a quarter of the Talbot distance, an image of the grating with half the period of the original grating forms; at an eighth, the period of the image is a fourth of the original, and so on. The set of all fractional images within one Talbot distance is called the “Talbot carpet” [30]. Considering the inherent self-similarity in the Talbot effect, it is not a surprise that the Talbot carpet was later found to follow a fractal structure. Talbot carpets have more recently been simulated on computers, resulting in colorful, vibrant fractal patterns.

3.3 Experimental Setup

The collimation test instrument we created (Figure 6) is based in large part on those previously described by Ganesan and Venkateswarlu (1993) [31], Kothiyal and Sirohi (1987) [6] and Silva (1980) [32]. However, it is somewhat simpler in that the usual beam splitter was replaced by an offset angled mirror. This simplification increases the intensity of the resulting moiré fringe patterns, making them easier to view and record. In the
setup, a 150 micron pinhole is placed 8.0 centimeters away from a 633 nm HeNe laser light source, and lens L1 (focal length 25.4 mm) is placed 15.0 centimeters from the pinhole. The pinhole creates a clean Airy pattern the central part of which is similar to an ideal Gaussian beam, and the lens causes the pattern to rapidly diverge after first bringing it to a focus. The resulting expanding wavefront is collimated (made parallel) by lens L2, the collimating lens, which has a focal length of 330 mm. In order for the laser beam to be precisely collimated, the collimating lens must be placed so that its focus (very nearly) coincides with the focus of the first lens. If the collimating lens is moved closer to lens L1 the beam will diverge, and vice versa. Lens L2 was mounted on a carrier that could be moved along a rail. The position of the carrier, and hence of the lens, could be set and measured to an uncertainty of about 1.0 mm with the aid of a meter stick secured to the optical table. Note that the zero position of the ruler does not correspond to the position of the first lens.

The key element of the collimation test instrument is a 50 lines/inch Ronchi grating tilted about 3 degrees from the vertical and placed 1.0 meters from the collimating lens. (A Ronchi grating is a specific type of grating in which the period is equal to twice the width of the slits.) A retroreflector (right-angle prism) is placed at half the grating’s Talbot distance away from the grating.

Incident light passes through the grating and forms a self image as it hits the retroreflector, half the Talbot distance away. The self image is reflected twice, so that the returning light forms a self image of the grating reflected about the vertical. This oppositely tilted self image passes once more through the original
grating, creating moiré fringes. These fringes are directed to the viewing screen (a sheet of white paper) by the angled mirror, where they are photographed from the back side of the paper with an ordinary digital camera (Sony Mavica FD-73). The position of the collimating lens was varied from 41.5 cm to 45.4 cm in 1.0 mm steps, and the resulting moiré fringe patterns were recorded.

As collimation decreases, the radius of curvature of the wavefront also decreases and the wavefront becomes spherical, creating either a divergent or a convergent beam. If the wavefront is spherical rather than planar, it follows that the resulting Talbot self image will be either magnified or diminished in size. This magnification of the image of the grating, or lack thereof, is equivalent to placing two gratings of different period on top of each other. Furthermore, the self image is tilted with respect to the original grating, so that the resulting moiré fringes will similarly tilt as the period of the self image changes. Following a formula given by Creath and Wyant (1992) [18], the tilt of the moiré fringes can be used to determine the difference in period between the grating and the self image:

$$\tan \theta = \frac{\lambda_1 + \lambda_2}{\lambda_2 - \lambda_1} \tan \alpha$$

[13]

where $\theta$ is the angle the fringes make with the vertical axis, and all other variables are defined as they are in Eqns. [1] through [5]. When the incident light is perfectly collimated, the period of the self image will be equal to that of the original grating. In this case, the fringes produced will be horizontal, as we saw in Eqn. [8] earlier. As the beam moves away from collimation, the fringes will acquire a greater tilt from the horizontal; furthermore, fringes produced by divergent light will tilt in an opposite direction from fringes produced by convergent light. Thus, the tilt of the moiré fringes with respect to the horizontal indicates unequivocally whether the beam is collimated, convergent, or divergent.

3.4 Results

We took pictures of moiré fringes for 40 different distances of the collimating lens from the first lens. Figure 7 shows eight of these photographs. The photographs were transferred to a computer using a floppy disk for further analysis. The fringe angles in the images were measured using ImageJ [33], a Java image processing and analysis software. This program was chosen because it had a tool that allowed the user to draw a line along the fringe direction to measure the angle relative to a reference line. Five angle measurements were taken for each picture at different locations in the image, and the results were averaged. The standard
deviation of the five measurements was also calculated in order to get an idea of the variations within the set of five measurements. The data are given in Table 1. The results are summarized in Fig. 8, which shows a plot of fringe tilt from vertical versus distance from first lens. The errors in the plot are smaller than the symbol size. There seems to be a roughly linear relation between fringe tilt from the vertical and distance from the first lens. The reason for the bump in the data at around 43.2 centimeters is not immediately obvious and likely results from an unknown systematic error.

In order to ascertain that the position of the collimating lens measured in Fig. 8 truly matched the point of collimation, we measured beam width at four different distances from the laser for different positions of the collimating lens. Figure 9 shows this graph. From the results in Fig. 8, we know that the point of collimation (when the fringe tilt is 90 degrees) is when the position of the collimating lens is around 44 centimeters. The beam can be considered collimated when the beam width does not change significantly over a distance. Figure 9 indeed corroborates that between 44.0 centimeters and 44.5 centimeters, the beam width does not significantly change. Work is under way to interpret Fig. 8 in a more quantitative manner.

The final step of analysis was to model the instrument in MATHEMATICA using the following equations in order to generate a prediction of the expected tilt angle versus position. For the purposes of this model, the lens position is measured by its “defocus,” or distance with respect to the position that gives perfect collimation.

Creath and Wyant (1992) [18] showed that the tilt of fringes as a function of the period difference
between the grating and its self image is given by Eqn. [13], which may be rewritten as

\[ \lambda_2 = \lambda_1 \left( \frac{\tan \alpha + \tan \theta}{\tan \theta - \tan \alpha} \right) \]  

[14]

\( \lambda_1 \) is plugged into Eqn. [14], and \( \lambda_2 \), the period of the self-image of the grating, is found. \( \lambda_2 \) can be used
Figure 10: Graph of defocus ($\Delta f$) versus fringe tilt from horizontal. The blue circles represent the experimental data, and the purple squares represent the results of MATHEMATICA calculation. From the graph, it is apparent that the program matches the data very well.

to find the change in period between the grating and the self image ($\Delta d = \lambda_2 - \lambda_1$). Ganesan (1993) [31] showed that the difference in period between the grating and its self-image can be written as

$$\Delta d = \frac{d^3}{\lambda R}$$  \[15\]

where $d$ is the period of the grating, $\lambda$ is the wavelength of the incident light, and $R$ is the radius of curvature of the beam. From here, $R$, the wavefront curvature, is calculated. This value is used in the following equation, also from Ganesan (1993):

$$R = \frac{f^2}{\Delta f}$$  \[16\]

Here, $f$ is the focal length of the collimating lens and $\Delta f$ is the defocus.

Figure 10 compares the data taken in the lab with the theoretical predictions given by the MATHEMATICA program. The slopes of the best-fit lines for the experimental data and the predictions are $-0.057$ and $-0.054$, respectively. It is evident from the graph that the data match the predictions closely.
4 Summary and Future Research

In this work, we studied and modeled moiré patterns and one of their applications, collimation testing. We created a variety of moiré patterns by simulating periodic gratings using Mathematica; these simulations are good pedagogical tools due to their capability to allow user interaction. We also worked with a quasiperiodic grating based on the pattern of florets on a sunflower. In addition to being mathematically interesting, such patterns are also of use in studying quasicrystals. We presented results from an experiment to test a simplified collimation tester setup, which is an application of the moiré and Talbot effects. We measured the tilt of the moiré fringes as a function of the position of the collimating lens and compared it to theoretical expectations obtained from a simple Mathematica program that calculated the defocus of the collimating lens as a function of the fringe tilt. In the future, we will attempt to better understand what “collimation” means in terms of this experiment.

Creating moiré patterns by computer simulation is not the same as having a mathematical expression for them. In the future, we would like to better understand and extend previous work of this type, such as that of Creath and Wyant (1992) [18], Oster et al. (1964) [19], Amidror (2003) [22], and Gabrielyan (2002) [34]. We intend to relate the moiré patterns on elliptical gratings to the eccentricity of the overlapped ellipses. We further plan to observe and study the Glass patterns created by quasiperiodic gratings. We are especially interested in deriving the mathematics for moiré patterns created with quasiperiodic gratings, such as the sunflower floret patterns that were modeled. We are also interested in finding different kinds of quasiperiodic patterns to model.

Moiré patterns are more than a fascinating optical phenomenon; they also have a huge host of applications in the real world. They can aid in understanding topics that are at the forefront of science, such as graphene and quasicrystals, and provide simpler and faster methods of accomplishing certain tasks that are invaluable to optics research, such as collimation testing and focal length measurement. Understanding the moiré effect can help digital photographers to prevent aliasing in photographs and movies, and make more effective anti-counterfeit prevention measures. It is very likely that the moiré effect will find even more practical applications in the future.

In the future, we hope to explore further the mathematical intricacies of the Talbot effect. It is certainly evident that while the practical applications for the Talbot and moiré effects are boundless, from a theoretical perspective, both are, and will remain, a testament to the stunning beauty in mathematics.
Table 1: Fringe tilt from vertical (degrees)

<table>
<thead>
<tr>
<th>Picture #</th>
<th>Position L2 cm</th>
<th>Meas. 1 deg</th>
<th>Meas. 2 deg</th>
<th>Meas. 3 deg</th>
<th>Meas. 4 deg</th>
<th>Meas. 5 deg</th>
<th>Avg. Tilt deg</th>
<th>S.D.  deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>709</td>
<td>41.5</td>
<td>56.00</td>
<td>56.48</td>
<td>55.85</td>
<td>56.53</td>
<td>55.84</td>
<td>56.14</td>
<td>.339</td>
</tr>
<tr>
<td>710</td>
<td>41.6</td>
<td>56.37</td>
<td>56.55</td>
<td>56.50</td>
<td>56.61</td>
<td>56.39</td>
<td>56.48</td>
<td>.103</td>
</tr>
<tr>
<td>711</td>
<td>41.7</td>
<td>56.88</td>
<td>57.41</td>
<td>56.37</td>
<td>57.36</td>
<td>57.09</td>
<td>57.02</td>
<td>.423</td>
</tr>
<tr>
<td>712</td>
<td>41.8</td>
<td>58.22</td>
<td>58.78</td>
<td>58.13</td>
<td>57.94</td>
<td>57.37</td>
<td>58.09</td>
<td>.099</td>
</tr>
<tr>
<td>713</td>
<td>41.9</td>
<td>59.00</td>
<td>59.41</td>
<td>58.57</td>
<td>58.70</td>
<td>59.09</td>
<td>58.95</td>
<td>.432</td>
</tr>
<tr>
<td>714</td>
<td>42.0</td>
<td>61.00</td>
<td>61.00</td>
<td>60.69</td>
<td>61.06</td>
<td>60.97</td>
<td>61.01</td>
<td>.222</td>
</tr>
<tr>
<td>715</td>
<td>42.1</td>
<td>62.07</td>
<td>62.33</td>
<td>62.45</td>
<td>61.84</td>
<td>61.36</td>
<td>62.17</td>
<td>.236</td>
</tr>
<tr>
<td>716</td>
<td>42.2</td>
<td>63.43</td>
<td>63.07</td>
<td>63.59</td>
<td>63.38</td>
<td>63.97</td>
<td>63.49</td>
<td>.792</td>
</tr>
<tr>
<td>717</td>
<td>42.3</td>
<td>64.38</td>
<td>64.34</td>
<td>64.32</td>
<td>64.65</td>
<td>64.48</td>
<td>64.43</td>
<td>.136</td>
</tr>
<tr>
<td>718</td>
<td>42.4</td>
<td>66.20</td>
<td>66.38</td>
<td>65.93</td>
<td>66.08</td>
<td>66.04</td>
<td>66.13</td>
<td>.172</td>
</tr>
<tr>
<td>719</td>
<td>42.5</td>
<td>66.72</td>
<td>66.90</td>
<td>67.55</td>
<td>66.14</td>
<td>67.98</td>
<td>67.06</td>
<td>.720</td>
</tr>
<tr>
<td>720</td>
<td>42.6</td>
<td>68.20</td>
<td>68.57</td>
<td>69.14</td>
<td>68.41</td>
<td>69.04</td>
<td>68.67</td>
<td>.405</td>
</tr>
<tr>
<td>721</td>
<td>42.7</td>
<td>70.04</td>
<td>70.22</td>
<td>70.28</td>
<td>70.21</td>
<td>70.18</td>
<td>70.19</td>
<td>.089</td>
</tr>
<tr>
<td>722</td>
<td>42.8</td>
<td>71.23</td>
<td>72.08</td>
<td>71.86</td>
<td>72.88</td>
<td>72.79</td>
<td>72.77</td>
<td>1.827</td>
</tr>
<tr>
<td>723</td>
<td>42.9</td>
<td>73.11</td>
<td>73.85</td>
<td>73.48</td>
<td>73.64</td>
<td>73.69</td>
<td>73.56</td>
<td>.281</td>
</tr>
<tr>
<td>724</td>
<td>43.0</td>
<td>75.74</td>
<td>76.42</td>
<td>76.55</td>
<td>76.59</td>
<td>76.40</td>
<td>76.34</td>
<td>.345</td>
</tr>
<tr>
<td>725</td>
<td>43.1</td>
<td>80.90</td>
<td>80.34</td>
<td>81.91</td>
<td>89.97</td>
<td>81.77</td>
<td>81.18</td>
<td>.654</td>
</tr>
<tr>
<td>726</td>
<td>43.2</td>
<td>83.35</td>
<td>82.96</td>
<td>83.48</td>
<td>83.04</td>
<td>83.02</td>
<td>83.17</td>
<td>.230</td>
</tr>
<tr>
<td>727</td>
<td>43.3</td>
<td>84.90</td>
<td>84.98</td>
<td>84.57</td>
<td>84.42</td>
<td>84.84</td>
<td>84.74</td>
<td>.237</td>
</tr>
<tr>
<td>728</td>
<td>43.4</td>
<td>86.08</td>
<td>86.15</td>
<td>86.30</td>
<td>86.08</td>
<td>86.58</td>
<td>86.24</td>
<td>.211</td>
</tr>
<tr>
<td>729</td>
<td>43.5</td>
<td>86.18</td>
<td>85.88</td>
<td>86.07</td>
<td>86.25</td>
<td>85.45</td>
<td>85.97</td>
<td>.321</td>
</tr>
<tr>
<td>730</td>
<td>43.6</td>
<td>86.80</td>
<td>86.71</td>
<td>86.47</td>
<td>86.19</td>
<td>86.78</td>
<td>86.59</td>
<td>.260</td>
</tr>
<tr>
<td>731</td>
<td>43.7</td>
<td>88.39</td>
<td>87.05</td>
<td>87.42</td>
<td>87.06</td>
<td>87.15</td>
<td>87.41</td>
<td>.566</td>
</tr>
<tr>
<td>732</td>
<td>43.8</td>
<td>87.94</td>
<td>88.21</td>
<td>87.56</td>
<td>88.12</td>
<td>88.17</td>
<td>88.00</td>
<td>.267</td>
</tr>
<tr>
<td>733</td>
<td>43.9</td>
<td>89.29</td>
<td>89.09</td>
<td>88.99</td>
<td>89.19</td>
<td>89.04</td>
<td>89.10</td>
<td>.132</td>
</tr>
<tr>
<td>734</td>
<td>44.0</td>
<td>91.65</td>
<td>91.53</td>
<td>91.64</td>
<td>91.33</td>
<td>91.59</td>
<td>91.55</td>
<td>.131</td>
</tr>
<tr>
<td>735</td>
<td>44.1</td>
<td>93.42</td>
<td>93.77</td>
<td>93.37</td>
<td>93.78</td>
<td>93.21</td>
<td>93.51</td>
<td>.254</td>
</tr>
<tr>
<td>736</td>
<td>44.2</td>
<td>94.37</td>
<td>94.21</td>
<td>94.42</td>
<td>94.33</td>
<td>94.49</td>
<td>94.36</td>
<td>.105</td>
</tr>
<tr>
<td>737</td>
<td>44.3</td>
<td>95.35</td>
<td>95.54</td>
<td>95.35</td>
<td>95.26</td>
<td>95.05</td>
<td>95.03</td>
<td>.629</td>
</tr>
<tr>
<td>738</td>
<td>44.4</td>
<td>96.05</td>
<td>96.84</td>
<td>96.97</td>
<td>97.09</td>
<td>96.28</td>
<td>96.64</td>
<td>.455</td>
</tr>
<tr>
<td>739</td>
<td>44.5</td>
<td>99.52</td>
<td>100.81</td>
<td>99.93</td>
<td>99.95</td>
<td>100.12</td>
<td>100.07</td>
<td>.471</td>
</tr>
<tr>
<td>740</td>
<td>44.6</td>
<td>100.56</td>
<td>101.16</td>
<td>101.47</td>
<td>101.48</td>
<td>101.67</td>
<td>101.27</td>
<td>.436</td>
</tr>
<tr>
<td>741</td>
<td>44.7</td>
<td>102.53</td>
<td>102.46</td>
<td>103.58</td>
<td>103.01</td>
<td>103.67</td>
<td>103.05</td>
<td>.567</td>
</tr>
<tr>
<td>742</td>
<td>44.8</td>
<td>105.86</td>
<td>105.13</td>
<td>105.84</td>
<td>106.06</td>
<td>106.22</td>
<td>105.82</td>
<td>.417</td>
</tr>
<tr>
<td>743</td>
<td>44.9</td>
<td>108.99</td>
<td>108.74</td>
<td>108.78</td>
<td>109.14</td>
<td>108.95</td>
<td>108.92</td>
<td>.163</td>
</tr>
<tr>
<td>744</td>
<td>45.0</td>
<td>111.14</td>
<td>111.60</td>
<td>111.67</td>
<td>112.13</td>
<td>111.96</td>
<td>111.76</td>
<td>.380</td>
</tr>
<tr>
<td>745</td>
<td>45.1</td>
<td>114.30</td>
<td>114.76</td>
<td>114.84</td>
<td>114.90</td>
<td>115.06</td>
<td>114.77</td>
<td>.286</td>
</tr>
<tr>
<td>746</td>
<td>45.2</td>
<td>116.92</td>
<td>116.57</td>
<td>117.49</td>
<td>116.87</td>
<td>117.10</td>
<td>116.99</td>
<td>.338</td>
</tr>
<tr>
<td>747</td>
<td>45.3</td>
<td>125.89</td>
<td>125.26</td>
<td>124.97</td>
<td>125.88</td>
<td>125.49</td>
<td>125.50</td>
<td>.398</td>
</tr>
<tr>
<td>748</td>
<td>45.4</td>
<td>128.04</td>
<td>128.28</td>
<td>128.14</td>
<td>128.09</td>
<td>127.59</td>
<td>128.03</td>
<td>.261</td>
</tr>
</tbody>
</table>

a Five measurements of angle were taken. b Standard deviation. See text for details.
References


[16] Dr. Larry Thibos, Indiana University (private communication).


[27] Lord Rayleigh, “On copying diffraction-gratings, and on some phenomena connected therewith, Phil. Mag., 11, 196 (1881).